

# Generalized Curvature Condition for Subelliptic Diffusion Processes\*

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## Abstract

By using a general version of curvature condition, derivative inequalities are established for a large class of subelliptic diffusion semigroups. As applications, the Harnack/cost-entropy/cost-variance inequalities for the diffusion semigroups, and the Poincaré/log-Sobolev inequalities for the associated Dirichlet forms in the symmetric case, are derived. Our results largely generalize and partly improve the corresponding ones obtained recently in [5].

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## 1 Introduction

It is well known that Bakry-Emery's curvature and curvature-dimension conditions have played crucial roles in the study of elliptic diffusion processes. When the diffusion operator is merely subelliptic, this condition is however no longer available. Recently, in order to study subelliptic diffusion processes, a nice generalized curvature-dimension condition was introduced and applied in [5, 6, 7], so that many important results derived in the elliptic setting have been extended to subelliptic diffusion processes with generators of type

$$L := \sum_{i=1}^n X_i^2 + X_0$$

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for smooth vector fields  $\{X_i : 0 \leq n \leq 1\}$  on a differential manifold such that  $\{X_i, \nabla_{X_i} X_j : 1 \leq i \leq n\}$  spans the tangent space (see comment (a) below). See also [3, 8, 9, 12] and the references within for the study of the heat semigroup generated by the Kohn-Laplacian on Heisenberg groups by other means. Stimulated by [5], in this paper we aim to introduce a new generalized curvature condition to study more general subelliptic diffusion processes.

Let  $M$  be a connected differential manifold, and let  $L$  be given above for some  $C^2$ -vector fields  $\{X_i\}_{i=1}^d$  and a  $C^1$ -vector field  $X_0$ . The square field (or carré du champ) for  $L$  is a symmetric bilinear differential form given by

$$\Gamma(f, g) = \sum_{i=1}^n (X_i f)(X_i g), \quad f, g \in C^1(M).$$

Obviously,  $\Gamma$  satisfies

$$\Gamma(f) := \Gamma(f, f) \geq 0, \quad \Gamma(fg, h) = g\Gamma(f, h) + f\Gamma(g, h), \quad \Gamma(\phi \circ f, g) = (\phi' \circ f)\Gamma(f, g)$$

for any  $f, g, h \in C^1(M)$  and  $\phi \in C^1(\mathbb{R})$ . From now on, a symmetric bilinear differential form  $\bar{\Gamma}$  satisfying these properties is called a diffusion square field. If moreover for any  $x \in M$  and  $f \in C^1(M)$ ,  $\bar{\Gamma}(f)(x) = 0$  implies  $(df)(x) = 0$ , we call  $\bar{\Gamma}$  elliptic or non-degenerate.

For any  $C^2$ -diffusion square field  $\bar{\Gamma}$  (i.e.  $\bar{\Gamma}(f, g) \in C^2(M)$  for  $f, g \in C^\infty(M)$ ), we define the associated Bakry-Emery curvature operator w.r.t.  $L$  by

$$\bar{\Gamma}_2(f) = \frac{1}{2}L\bar{\Gamma}(f) - \bar{\Gamma}(f, Lf), \quad f \in C^3(M).$$

Then the generalized curvature-dimension condition introduced in [7] reads

$$(1.1) \quad \Gamma_2(f) + r\Gamma_2^Z(f) \geq \frac{(Lf)^2}{d} + \left(\rho_1 - \frac{\kappa}{r}\right)\Gamma(f) + \rho_2\Gamma^Z(f), \quad f \in C^2(M), r > 0,$$

where  $\rho_2 > 0, \kappa \geq 0, \rho_1 \in \mathbb{R}$  and  $d \in (0, \infty]$  are constants, and  $\Gamma^Z$  is a  $C^2$ -diffusion square field such that  $\Gamma + \Gamma^Z$  is elliptic and

$$(1.2) \quad \Gamma(\Gamma^Z(f), f) = \Gamma^Z(\Gamma(f), f), \quad f \in C^\infty(M)$$

holds. When  $\Gamma^Z = 0$ , (1.1) reduces back to the Bakry-Emery curvature-dimension condition [2], and when  $d = \infty$  it becomes the following generalized curvature condition

$$(1.3) \quad \Gamma_2(f) + r\Gamma^Z(f) \geq \left(\rho_1 - \frac{\kappa}{r}\right)\Gamma(f) + \rho_2\Gamma^Z(f), \quad f \in C^2(M), r > 0.$$

Using (1.2) and (1.3) for symmetric subelliptic operators, Poincaré inequality for the associated Dirichlet form, and the Harnack inequality and the log-Sobolev inequality (for, however, an enlarged Dirichlet form given by  $\Gamma + \Gamma^Z$ ) for the associated diffusion semigroup, and the HWI inequality (where the energy part is given by the enlarged Dirichlet form) are investigated in [5].

The main purpose of this paper is to introduce a general version of the curvature condition to derive better inequalities for more general subelliptic diffusion semigroups. The necessity of our study is based on the following three observations:

- (a) Condition (1.3) is not available if the family  $\{X_i, \nabla_{X_i} X_j\}_{1 \leq i, j \leq n}$  does not span the tangent space  $TM$ , where  $\nabla$  is the Levi-Civita connection w.r.t. a Riemannian metric. Indeed, (1.3) implies that the diffusion square field

$$\tilde{\Gamma}(f) := \Gamma(f) + \sum_{i,j=1}^n ((\nabla_{X_i} X_j)f)^2, \quad f \in C^1(M)$$

is elliptic. To see this, let  $x \in M$  and  $f \in C^1(M)$  such that  $\tilde{\Gamma}(f)(x) = 0$ . Applying (1.3) and letting  $r \rightarrow 0$  we obtain  $\Gamma_2(f)(x) \geq \rho_2 \Gamma^Z(f)(x)$ . Assuming further that  $\text{Hess}_f(x) = 0$  (since one may find such a function  $\tilde{f}$  with  $df(x) = d\tilde{f}(x)$ ), it is easy to see from  $\Gamma_2(f)(x) \geq \rho_2 \Gamma^Z(f)(x)$  and  $\tilde{\Gamma}(f)(x) = 0$  that  $\Gamma^Z(f)(x) = 0$ . Since  $\Gamma + \Gamma^Z$  is elliptic, we conclude that  $\tilde{\Gamma}(f)(x) = 0$  implies  $df(x) = 0$ , so that  $\tilde{\Gamma}$  is elliptic as well.

- (b) Even when the family  $\{X_i, \nabla_{X_i} X_j\}_{1 \leq i, j \leq n}$  spans the tangent space  $TM$ , (1.3) may hold for some functions of  $r$  in place of  $\rho_1 - \frac{\kappa}{r}$  and  $\rho_2$  therein (see Subsection 4.1).
- (c) Combining back to (1.3), recall that under condition (1.2) it was proved in [5] (see Proposition 3.1 therein) that

$$(1.4) \quad \Gamma(P_t f) + \rho_2 t \Gamma^Z(P_t f) \leq \left( \frac{\rho_2 + 2\kappa}{\rho_2 t} + 2\rho_1^- \right) (P_t f) \{P_t(f \log f) - (P_t f) \log P_t f\}$$

holds for all  $t > 0$  and positive  $f \in \mathcal{B}_b(M)$ . Comparing with known sharp gradient inequality for the elliptic case, i.e. for  $\kappa = 0$  and  $\Gamma^Z = 0$  one has

$$\Gamma(P_t f) \leq \frac{2\rho_1}{e^{2\rho_1 t} - 1} (P_t f) \{P_t(f \log f) - (P_t f) \log P_t f\},$$

where  $\frac{2\rho_1}{e^{2\rho_1 t} - 1} := \frac{1}{t}$  for  $\rho_1 = 0$ , the inequality (1.4) is less sharp when  $\rho_1 \neq 0$ . So, it would be nice to find an exact extension of this sharp inequality to the subelliptic setting (see Proposition 3.6 below).

The generalized curvature-dimension condition we proposed here is

$$\Gamma_2(f) + \sum_{i=1}^l r_i \Gamma_2^{(i)}(f) \geq \frac{(Lf)^2}{d} + \sum_{i=0}^l K_i(r_1, \dots, r_l) \Gamma^{(i)}(f), \quad f \in C^3(M), r_1, \dots, r_l > 0,$$

where  $d \in (0, \infty]$  is a constant,  $\Gamma^{(0)} := \Gamma$ ,  $\{\Gamma^{(i)}\}_{1 \leq i \leq l}$  are some  $C^2$ -diffusion square fields, and  $\{K_i\}_{0 \leq i \leq l}$  are some continuous functions on  $(0, \infty)^l$ . In this paper, we will only consider the condition with  $d = \infty$ , i.e.

$$(1.5) \quad \Gamma_2(f) + \sum_{i=1}^l r_i \Gamma_2^{(i)}(f) \geq \sum_{i=0}^l K_i(r_1, \dots, r_l) \Gamma^{(i)}(f), \quad f \in C^3(M), r_1, \dots, r_l > 0,$$

but the condition with finite  $d$  will be useful for other purposes as in [6, 7]. In fact, we will make use of the following assumption.

(A) (1.5) holds for some  $C^2$ -diffusion square fields  $\{\Gamma^{(i)}\}_{i=0}^l$  and  $\{K_i\}_{0 \leq i \leq l} \subset C((0, \infty)^l)$ , where  $\Gamma^{(0)} = \Gamma$ . There exists a smooth compact function  $W \geq 1$  on  $M$  and a constant  $C > 0$  such that  $LW \leq CW$  and  $\tilde{\Gamma}(W) \leq CW^2$ , where  $\tilde{\Gamma} = \sum_{i=0}^l \Gamma^{(i)}$ .

Recall that  $W$  is called a compact function if  $\{W \leq r\}$  is compact for any constant  $r$ . The condition  $LW \leq CW$  is standard to ensure the non-explosion of the  $L$ -diffusion process, and the condition  $\tilde{\Gamma}(W) \leq CW^2$  is used to prove the boundedness of  $\tilde{\Gamma}(P_t f)$  for  $f \in \mathcal{C}$ , where

$$\mathcal{C} := \left\{ f \in C^\infty(M) \cap \mathcal{B}_b(M) : \tilde{\Gamma}(f) \text{ is bounded} \right\}.$$

We note that under (1.3) the boundedness of  $\tilde{\Gamma}(P_t f)$  is claimed in [7] using a global parabolic comparison theorem, which, however, is not yet available on general non-compact manifolds.

Similarly to the analysis of elliptic diffusions, a starting point for analyzing the semigroup using a curvature condition is the following “gradient” inequalities, which generalize the corresponding ones derived in [5]. Let

**Theorem 1.1.** *Assume (A). For fixed  $t > 0$ , let  $\{b_i\}_{0 \leq i \leq l} \subset C^1([0, t])$  be strictly positive on  $(0, t)$  such that*

$$(1.6) \quad b'_i(s) + 2 \left\{ b_0 K_i \left( \frac{b_1}{b_0}, \dots, \frac{b_l}{b_0} \right) \right\}(s) \geq 0, \quad s \in (0, t), 1 \leq i \leq l$$

and

$$c_b := - \inf_{(0, t)} \left\{ b'_0 + 2b_0 K_0 \left( \frac{b_1}{b_0}, \dots, \frac{b_l}{b_0} \right) \right\} < \infty.$$

Then:

(1) For any  $f \in \mathcal{C}$ ,

$$2 \sum_{i=0}^l \left\{ b_i(0) \Gamma^{(i)}(P_t f) - b_i(t) P_t \Gamma^{(i)}(f) \right\} \leq c_b \{ P_t f^2 - (P_t f)^2 \}.$$

(2) If

$$(1.7) \quad \Gamma^{(i)}(\Gamma(f), f) = \Gamma(\Gamma^{(i)}(f), f), \quad 1 \leq i \leq l, f \in C^\infty(M),$$

then for any positive  $f \in \mathcal{C}$ ,

$$\sum_{i=0}^l \left\{ b_i(0) \frac{\Gamma^{(i)}(P_t f)}{P_t f} - b_i(t) P_t \frac{\Gamma^{(i)}(f)}{f} \right\} \leq c_b \{ P_t (f \log f) - (P_t f) \log P_t f \}.$$

Theorem 1.1 will be proved in Section 2. In Section 3 we apply this theorem to the study of “gradient” estimate, Poincaré inequality, Harnack inequality and applications. In Section 4 we present some specific examples to illustrate the generalized curvature condition (1.5),

for which (1.3) does not hold. We will not consider the log-Sobolev and HWI inequalities for enlarged Dirichlet forms given by  $\sum_{i=0}^l \Gamma^{(i)}$ , since they are no longer intrinsic for the underlying subelliptic diffusion process. It is still open to derive the intrinsic semigroup log-Sobolev inequality and HWI inequality for subelliptic diffusion processes using generalized curvature conditions.

Finally, noting that (1.5) is still not available for some highly degenerate subelliptic diffusion operators, e.g.  $L = \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial y}$  on  $\mathbb{R}^2$  for which the derivative formula and Harnack ineuqlity has been established in [19, 11], we propose in Section 5 an extension of Theorem 1.1 with (1.5) holding for not necessarily positively definite  $\{\Gamma^{(i)}\}_{1 \leq i \leq l}$  and not necessarily all  $r_i > 0$ . As application, explicit gradient-entropy inequality and Harnack inequality are presented for this simple example.

## 2 Proof of Theorem 1.1

To prove this theorem using a modified Bakry-Emery semigroup argument as in [5], we need to first confirm that  $P_t \mathcal{C} \subset \mathcal{C}$ , which follows immediately from the following lemma.

**Lemma 2.1.** *Assume (A) and let  $K = \min_{0 \leq i \leq l} K_i(1, \dots, 1)$ . Then*

$$(2.1) \quad \tilde{\Gamma}(P_t f) \leq e^{-2Kt} P_t \tilde{\Gamma}(f), \quad t \geq 0, f \in \mathcal{C}.$$

*Proof.* (i) We first prove for any  $f \in C_0^2(M)$  and  $t > 0$ ,  $\tilde{\Gamma}(P_t f)$  is bounded on  $[0, t] \times M$ . To this end, we approximate the generator  $L$  by using operators with compact support, so that the approximating diffusion processes stay in compact sets. Take  $h \in C_0^\infty([0, \infty))$  such that  $h' \leq 0, h|_{[0,1]} = 1$  and  $\text{supp} h = [0, 2]$ . For any  $m \geq 1$ , let  $\varphi_m = h(W/m)$  and  $L_m = \varphi_m^2 L$ . Then  $L_m$  has compact support  $B_m := \{W \leq 2m\}$ . Let  $x \in \{W \leq m\}$  and  $X_s^m$  be the  $L_m$ -diffusion process starting at  $x$ . Let

$$\tau_m = \inf\{s \geq 0 : W(X_s^m) \geq 2m\}.$$

Since  $LW \leq CW, \Gamma(W) \leq \tilde{\Gamma}(W) \leq CW^2, h' \leq 0, 0 \leq h \leq 1$  and  $h'(W/m) = 0$  for  $W \geq 2m$ , we have

$$\begin{aligned} L_m \frac{1}{\varphi_m^2} &= -\frac{2L\varphi_m}{\varphi_m} + \frac{6\Gamma(\varphi_m)}{\varphi_m^2} \\ &= -\frac{2h'(W/m)LW}{m\varphi_m} - \frac{2h''(W/m)\Gamma(W)}{m^2\varphi_m} + \frac{6h'(W/m)^2\Gamma(W)}{m^2\varphi_m^2} \leq \frac{C_1}{\varphi_m^2} \end{aligned}$$

for some constant  $C_1 > 0$  independent of  $m$ . By a standard argument, this implies that  $\tau_m = \infty$  and

$$(2.2) \quad \mathbb{E}\left(\frac{1}{\varphi_m^2}\right)(X_s) \leq \frac{e^{C_1 t}}{\varphi_m^2(x)} = e^{C_1 s}, \quad s \geq 0.$$

Now, let  $P_s^m$  be the diffusion semigroup generated by  $L_m$ . By the Itô formula and  $\tilde{\Gamma}_2 \geq K\tilde{\Gamma}$  implied by **(A)** we obtain

$$\begin{aligned}
(2.3) \quad d\tilde{\Gamma}(P_{t-s}^m f)(X_s^m) &= dM_s^m + \{\varphi_m^2 L\tilde{\Gamma}(P_{t-s}^m f) - 2\tilde{\Gamma}(P_{t-s}^m f, \varphi_m^2 L P_{t-s}^m f)\}(X_s^m)ds \\
&\geq dM_s^m + \{2\varphi_m^2 \tilde{\Gamma}_2(P_{t-s}^m f) - 4\tilde{\Gamma}(\log \varphi_m, P_{t-s}^m f)P_{t-s}^m L_m f\}(X_s^m)ds \\
&\geq dM_s^m - \left\{2|K|\tilde{\Gamma}(P_{t-s}^m f) + 4\|Lf\|_\infty \sqrt{\tilde{\Gamma}(\log \varphi_m)\tilde{\Gamma}(P_{t-s}^m f)}\right\}(X_s^m)ds \\
&\geq dM_s^m - C_2 \tilde{\Gamma}(P_{t-s}^m f)(X_s^m)ds - \tilde{\Gamma}(\log \varphi_m)(X_s^m)ds, \quad s \in [0, t]
\end{aligned}$$

for some martingale  $M_s^m$  and some constant  $C_2 > 0$  independent of  $m$ . Since  $h'(W/m) = 0$  for  $W \geq 2m$  and  $\tilde{\Gamma}(W) \leq CW^2$ ,

$$\tilde{\Gamma}(\log \varphi_m) = \frac{h'(W/m)^2 \tilde{\Gamma}(W)}{m^2 \varphi_m^2} \leq \frac{C_3}{\varphi_m^2}$$

holds for some constant  $C_3 > 0$  independent of  $m$ . Combining this with (2.2) and (2.3) we conclude that

$$(2.4) \quad \tilde{\Gamma}(P_t^m f) \leq e^{C_2 t} P_t^m \tilde{\Gamma}(f) + C_3 \int_0^t \mathbb{E}\left(\frac{1}{\varphi_m^2}\right)(X_s^m)ds \leq e^{C_2 t} \|\tilde{\Gamma}(f)\|_\infty + C_3 e^{C_1 t}$$

holds on  $\{W \leq m\}$ . Letting  $\tilde{\rho}$  be the intrinsic distance induced by  $\tilde{\Gamma}$ , i.e.

$$\tilde{\rho}(z, y) := \sup\{|g(x) - g(y)| : \tilde{\Gamma}(g) \leq 1\}, \quad z, y \in M,$$

we deduce from (2.4) that for any  $x, y \in M$ ,

$$(2.5) \quad |P_t^m f(z) - P_t^m f(y)|^2 \leq \tilde{\rho}(z, y)^2 \left( e^{C_2 t} \|\tilde{\Gamma}(f)\|_\infty + C_3 e^{C_1 t} \right), \quad t > 0$$

holds for large enough  $m$ . Noting that the  $L$ -diffusion process is non-explosive and  $X_s^m$  is indeed generated by  $L$  before time  $\sigma_m := \inf\{s \geq 0 : W(X_s^m) \geq m\}$  which increases to  $\infty$  as  $m \rightarrow \infty$ , we conclude that  $\lim_{m \rightarrow \infty} P_t^m f = P_t f$  holds point-wisely. Therefore, letting  $m \rightarrow \infty$  in (2.5) we obtain

$$|P_t f(z) - P_t f(y)|^2 \leq \tilde{\rho}(z, y)^2 \left( e^{C_2 t} \|\tilde{\Gamma}(f)\|_\infty + C_3 e^{C_1 t} \right), \quad t \geq 0, y, z \in M.$$

This implies that  $\tilde{\Gamma}(P_t f)$  is bounded on  $[0, t] \times M$  for any  $t > 0$ .

(ii) By an approximation argument, it suffices to prove (2.1) for  $f \in C_0^2(M)$ . By the Itô formula and (1.5), there exists a local martingale  $M_s$  such that

$$d\tilde{\Gamma}(P_{t-s} f)(X_s) = dM_s + 2\tilde{\Gamma}_2(P_{t-s} f)(X_s)ds \geq dM_s + 2K\tilde{\Gamma}(P_{t-s} f)(X_s)ds, \quad s \in [0, t].$$

Thus,

$$[0, t] \ni s \mapsto e^{-2Ks} \tilde{\Gamma}(P_{t-s} f)(X_s)$$

is a local submartingale. Since due to (a) this process is bounded, so that it is indeed a submartingale. Therefore, (2.1) holds.  $\square$

*Proof of Theorem 1.1.* (1) It suffices to prove for  $f \in C^\infty(M)$  which is constant outside a compact set. In this case we have  $\frac{d}{ds}P_s f = LP_s f = P_s Lf$ . Since  $X_s$  is non-explosive, by the Itô formula for any  $0 \leq i \leq l$  there exists a local martingale  $M_s^{(i)}$  such that

$$\begin{aligned} d\Gamma^{(i)}(P_{t-s}f)(X_s) &= dM_s^{(i)} + \{L\Gamma^{(i)}(P_{t-s}f) - 2\Gamma^{(i)}(P_{t-s}f, LP_{t-s}f)\}(X_s)ds \\ &= dM_s^{(i)} + 2\Gamma_2^{(i)}(P_{t-s}f)(X_s)ds, \quad s \in [0, t]. \end{aligned}$$

Therefore, due to (1.5) and (1.6), there exists a local martingale  $M_s$  such that

$$\begin{aligned} & d\left\{\sum_{i=0}^l b_i(s)\Gamma^{(i)}(P_{t-s}f)(X_s)\right\} \\ & \geq dM_s + \left\{\sum_{i=0}^l \left(2b_i(s)\Gamma_2^{(i)}(P_{t-s}f) + b'_i(s)\Gamma^{(i)}(P_{t-s}f)\right)\right\}(X_s)ds \\ & \geq dM_s + \left\{\sum_{i=0}^l \left(b'_i(s) + 2b_0(s)K_i\left(\frac{b_1}{b_0}, \dots, \frac{b_l}{b_0}\right)(s)\right)\Gamma^{(i)}(P_{t-s}f)\right\}(X_s)ds \\ & \geq dM_s + \left\{b'_0(s) + 2b_0(s)K_0\left(\frac{b_1}{b_0}, \dots, \frac{b_l}{b_0}\right)(s)\right\}\Gamma(P_{t-s}f)(X_s)ds. \end{aligned}$$

So, if  $c_b < \infty$  then

$$\sum_{i=0}^l b_i(s)\Gamma^{(i)}(P_{t-s}f)(X_s) + c_b \int_0^s \Gamma(P_{t-r}f)(X_r)dr$$

is a local submartingale for  $s \in [0, t]$ . Since, due to Lemma 2.1,  $\{\Gamma^{(i)}(P_{t-s}f)\}_{0 \leq i \leq l}$  are bounded, it is indeed a submartingale. In particular,

$$\sum_{i=0}^l \{b_i(0)\Gamma^{(i)}(P_t f) - b_i(t)P_t \Gamma^{(i)}(f)\} \leq c_b \int_0^t P_s \Gamma(P_{t-s}f)ds.$$

Then the proof is finished by noting that

$$P_s \Gamma(P_{t-s}f) = \frac{1}{2} \frac{d}{ds} P_s (P_{t-s}f)^2.$$

(2) Let  $f$  be strictly positive and be constant outside a compact set. Let

$$\phi^{(i)}(s, x) = \{(P_{t-s}f)\Gamma^{(i)}(\log P_{t-s}f)\}(x), \quad 0 \leq i \leq l, s \in [0, t], x \in M.$$

It is easy to see that (1.7) implies (cf. [7])

$$L\phi^{(i)} + \frac{\partial}{\partial s}\phi^{(i)} = 2(P_{t-s}f)\Gamma_2^{(i)}(\log P_{t-s}f), \quad 0 \leq i \leq l.$$

So, for each  $0 \leq i \leq l$ , there exists a local martingale  $M_s^{(i)}$  such that

$$d\phi^{(i)}(s, X_s) = dM_s^{(i)} + 2\{(P_{t-s}f)\Gamma_2^{(i)}(\log P_{t-s}f)\}(X_s)ds, \quad s \in [0, t].$$

The remainder of the proof is then completely similar to (1); that is,

$$\sum_{i=0}^l b_i(s) \{(P_{t-s}f)\Gamma^{(i)}(\log P_{t-s}f)\}(X_s) + c_b \int_0^s \{(P_{t-r}f)\Gamma(\log P_{t-r}f)\}(X_r)dr$$

is a submartingale for  $s \in [0, t]$ , so that the desired inequality follows by noting that

$$P_s \{(P_{t-s}f)\Gamma(\log P_{t-s}f)\} = \frac{d}{ds} P_s \{(P_{t-s}f) \log P_{t-s}f\}.$$

□

### 3 Applications of Theorem 1.1

For any non-negative symmetric measurable functions  $\tilde{\rho}$  on  $M \times M$ , let  $W_2^{\tilde{\rho}}$  be the  $L^2$ -transportation-cost with cost function  $\tilde{\rho}$ ; i.e. for any two probability measures  $\mu_1, \mu_2$  on  $M$ ,

$$W_2^{\tilde{\rho}}(\mu_1, \mu_2) := \inf_{\pi \in \mathcal{C}(\mu_1, \mu_2)} \pi(\tilde{\rho}^2)^{1/2},$$

where  $\pi(\tilde{\rho})$  stands for the integral of  $\tilde{\rho}$  w.r.t.  $\pi$ , and  $\mathcal{C}(\mu_1, \mu_2)$  is the set of all couplings of  $\mu_1$  and  $\mu_2$ .

#### 3.1 $L^2$ -derivative estimate and applications

**Proposition 3.1.** *Assume (A). Let  $t > 0$  and  $\{b_i\}_{0 \leq i \leq l} \subset C^1([0, t])$  be strictly positive in  $(0, t)$  such that (1.6) holds. If  $b_i(t) = 0, 0 \leq i \leq l$  and  $c_b < \infty$ , then:*

(1) *For any  $f \in \mathcal{B}_b(M)$ ,*

$$(3.1) \quad \sum_{i=0}^l b_i(0) \Gamma^{(i)}(P_t f) \leq c_b \{P_t f^2 - (P_t f)^2\}.$$

(2) *For any non-negative  $f \in \mathcal{B}_b(M)$ , the Harnack type inequality*

$$(3.2) \quad P_t f(x) \leq P_t f(y) + c_b \rho_b(x, y) \sqrt{P_t f^2(x)}, \quad x, y \in M$$

*holds for  $\rho_b$  being the intrinsic distance induced by  $\Gamma_b := \sum_{i=0}^l b_i(0) \Gamma^{(i)}$ .*



(3) If  $P_t$  has an invariant probability measure  $\mu$ , then for any  $f \geq 0$  with  $\mu(f) = 1$ , the variance-cost inequality

$$(3.3) \quad \text{Var}_\mu(P_t^* f) \leq c_b W_2^{\rho_b}(f\mu, \mu) \sqrt{\mu((P_t^* f)^3)}$$

holds, where  $P_t^*$  is the adjoint operator of  $P_t$  in  $L^2(\mu)$ , and

$$\text{Var}_\mu(P_t^* f) := \mu((P_t^* f)^2) - \mu(P_t^* f)^2 = \mu((P_t^* f)^2) - 1.$$

*Proof.* The first assertion is a direct consequence of Theorem 1.1(1), while according to [16, Proposition 1.3] (3.1) is equivalent to (3.2). Finally, (3.3) follows from (3.2) according to the following lemma 3.2.  $\square$

**Lemma 3.2.** *Let  $P$  be a Markov operator on  $\mathcal{B}_b(E)$  for a measurable space  $(E, \mathcal{B})$ . Let  $\mu$  be an invariant probability measure of  $P$ . If*

$$(3.4) \quad Pf(x) \leq Pf(y) + C\rho(x, y)\sqrt{Pf^2(x)}, \quad f \in \mathcal{B}_b^+(E)$$

holds for some constant  $C > 0$  and non-negative symmetric function  $\rho$  on  $E \times E$ , then

$$\text{Var}_\mu(P^* f) \leq CW_2^\rho(f\mu, \mu) \sqrt{\mu((P^* f)^3)}, \quad f \geq 0, \mu(f) = 1.$$

*Proof.* Let  $f \geq 0$  with  $\mu(f) = 1$ . For any  $\pi \in \mathcal{C}(f\mu, \mu)$ , (3.4) implies

$$\begin{aligned} \mu((P^* f)^2) &= \mu(fPP^* f) = \int_{E \times E} P(P^* f)(x) \pi(dx, dy) \\ &\leq \int_{E \times E} P(P^* f)(y) \pi(dx, dy) + C \int_{E \times E} \rho(x, y) \sqrt{P(P^* f)^2(x)} \pi(dx, dy) \\ &\leq \mu(PP^* f) + C \sqrt{\pi(\rho^2) \mu(fP(P^* f)^2)} = 1 + C \sqrt{\pi(\rho^2) \mu((P^* f)^3)}. \end{aligned}$$

This completes the proof.  $\square$

## 3.2 Entropy-derivative estimate and applications

**Proposition 3.3.** *Assume (A) and (1.7). Let  $t > 0$  and  $\{b_i\}_{0 \leq i \leq l} \subset C^1([0, t])$  be strictly positive in  $(0, t)$  such that (1.6) holds. If  $b_i(t) = 0, 0 \leq i \leq l$  and  $c_b < \infty$ , then:*

(1) For any  $f \in \mathcal{B}_b(M)$ ,

$$(3.5) \quad \sum_{i=0}^l b_i(0) \Gamma^{(i)}(P_t f) \leq c_b(P_t f) \{P_t(f \log f) - (P_t f) \log P_t f\}.$$

(2) For any non-negative  $f \in \mathcal{B}_b(M)$  and  $\alpha > 1$ , the Harnack type inequality

$$(3.6) \quad (P_t f)^\alpha(x) \leq P_t f^\alpha(y) \exp \left[ \frac{\alpha c_b \rho_b(x, y)^2}{4(\alpha - 1)} \right], \quad x, y \in M$$

holds for  $\rho_b$  being the intrinsic distance induced by  $\Gamma_b := \sum_{i=0}^l b_i(0)\Gamma^{(i)}$ . Consequently, the log-Harnack inequality

$$(3.7) \quad P_t \log f(x) \leq \log P_t f(y) + \frac{c_b \rho_b(x, y)^2}{4}$$

holds for uniformly positive measurable function  $f$ .

- (3) If  $P_t$  has an invariant probability measure  $\mu$ , then for any  $f \geq 0$  with  $\mu(f) = 1$ , the entropy-cost inequality

$$(3.8) \quad \mu((P_t^* f) \log P_t^* f) \leq \frac{c_b^2}{4} W_2^{\rho_b}(f\mu, \mu)^2.$$

*Proof.* The first assertion is a direct consequence of Theorem 1.1(2), (3.6) follows from (1) and the following Lemma 3.4, (3.7) follows from (3.6) according to [15, Proposition 2.2], and finally, (3.8) follows from (3.7) according to [17, Proposition 4.6].  $\square$

**Lemma 3.4.** *Let  $P$  be a Markov operator on  $\mathcal{B}_b(M)$  and let  $\gamma$  be a positive measurable function on  $(0, \infty)$ . Let  $\bar{\Gamma}$  be a smooth diffusion square field on  $E$  with intrinsic distance  $\bar{\rho}$ . If*

$$(3.9) \quad \sqrt{\bar{\Gamma}(Pf)} \leq \delta \{P(f \log f) - (Pf) \log Pf\} + \gamma(\delta)Pf, \quad \delta > 0$$

holds for all positive  $f \in \mathcal{B}_b(M)$ , then for any  $\alpha > 1$ ,

$$(Pf)^\alpha(x) \leq (Pf^\alpha(y)) \exp \left[ \int_0^1 \frac{\alpha \bar{\rho}(x, y)}{1 + (\alpha - 1)s} \gamma \left( \frac{\alpha - 1}{(1 + (\alpha - 1)s)\bar{\rho}(x, y)} \right) ds \right], \quad x, y \in M$$

holds for all positive  $f \in \mathcal{B}_b(M)$ .

*Proof.* The proof is completely similar to that of [1, Theorem 1.2]. Let  $\bar{\rho}(x, y) < \infty$ , and let  $x : [0, 1] \rightarrow M$  with constant speed w.r.t.  $\bar{\rho}$  such that  $x_0 = x, x_1 = y$ ; that is, the curve is the minimal geodesic from  $x$  to  $y$  induced by the metric associated to  $\bar{\Gamma}$ . Then  $|\frac{df(x_s)}{ds}|^2 \leq \bar{\rho}(x, y)^2 \bar{\Gamma}(f)(x_s)$  holds for all  $f \in C^1(M)$ . Let  $\beta(s) = 1 + (\alpha - 1)s$ . It follows from (3.9) that

$$\begin{aligned} & \frac{d}{ds} \log(Pf^{\beta(s)})^{\frac{\alpha}{\beta(s)}}(x_s) \\ & \geq \frac{\alpha(\alpha - 1) \{P(f^{\beta(s)} \log f^{\beta(s)}) - (Pf^{\beta(s)}) \log Pf^{\beta(s)}\}}{\beta(s)^2 Pf^{\beta(s)}}(x_s) - \frac{\alpha \bar{\rho}(x, y) \sqrt{\bar{\Gamma}(f)}}{\beta(s) Pf^{\beta(s)}}(x_s) \\ & \geq -\frac{\alpha \bar{\rho}(x, y)}{\beta(s)} \gamma \left( \frac{\alpha - 1}{\beta(s) \bar{\rho}(x, y)} \right). \end{aligned}$$

Then the proof is finished by integrating both sides on  $[0, 1]$  w.r.t.  $ds$ .  $\square$

For applications of the Harnack and log-Harnack inequalities, we refer to Section 4 of [17] (see also [15, 18]). In particular, if  $P_t$  is symmetric w.r.t. some probability measure  $\mu$  such that  $\mu(e^{\lambda \rho_b(o, \cdot)^2}) < \infty$  holds for some  $\lambda > \frac{c_b}{4}$ , then (3.6) implies that the log-Sobolev inequality

$$\mu(f^2 \log f^2) \leq c \mu(\Gamma(f)), \quad f \in C_0^1(M), \quad \mu(f^2) = 1$$

holds for some constant  $c > 0$ .

### 3.3 Exponential decay and Poincaré inequality

**Proposition 3.5.** *Assume (A). For  $r_i > 0, 1 \leq i \leq l$ , let*

$$\lambda(r_1, \dots, r_l) = \min_{0 \leq i \leq l} \frac{K_i(r_1, \dots, r_l)}{r_i},$$

where  $r_0 := 1$ . Then

$$\sum_{i=0}^l r_i \Gamma^{(i)}(P_t f) \leq e^{-2\lambda(r_1, \dots, r_l)t} \sum_{i=0}^l r_i P_t \Gamma^{(i)}(f), \quad t \geq 0, f \in C_b^1(M).$$

Consequently, if  $P_t$  is symmetric w.r.t a probability measure  $\mu$  and

$$\lambda := \sup_{r_1, \dots, r_l > 0} \lambda(r_1, \dots, r_l) > 0,$$

then the Poincaré inequality

$$(3.10) \quad \mu(f^2) \leq \frac{1}{\lambda} \mu(\Gamma(f)) + \mu(f)^2, \quad f \in C_0^1(M)$$

holds.

*Proof.* By a standard spectral theory (cf. the proof of [5, Corollary 2.4]), the Poincaré inequality follows immediately from the desired derivative inequality. To prove the derivative inequality, we take

$$b_0(s) = e^{-2\lambda(r_1, \dots, r_l)s}, \quad b_i(s) = r_i b_0(s), \quad 1 \leq i \leq l, s \geq 0.$$

Then

$$b'_i + 2b_0 K_i\left(\frac{b_1}{b_0}, \dots, \frac{b_l}{b_0}\right) = -2r_i \lambda(r_1, \dots, r_l) b_0 + 2b_0 K_i(r_1, \dots, r_l) \geq 0, \quad 0 \leq i \leq l.$$

Therefor, the desired gradient inequality follows from Theorem 1.1(1). □

When (1.3) holds for  $\rho_1, \rho_2 > 0$  and  $\kappa \geq 0$ , we have

$$\lambda = \sup_{r > 0} \left\{ \frac{\rho_2}{r} \wedge \left( \rho_1 - \frac{\kappa}{r} \right) \right\} = \frac{\rho_1 \rho_2}{\rho_2 + \kappa} > 0.$$

Thus, (3.10) recovers the Poincaré inequality presented in [5, Corollary 2.4].

### 3.4 Refined derivative inequalities under (1.3)

Coming back to condition (1.3), Theorem 1.1 implies the following exact extensions of sharp gradient estimates in the elliptic setting (see the above comment (c)).

**Proposition 3.6.** *Assume (1.3) for some constants  $\rho_2 > 0, \kappa \geq 0$  and  $\rho_1 \in \mathbb{R}$ . Assume there exist a smooth compact function  $W \geq 1$  and a constant  $C > 0$  such that  $LW \leq CW$  and  $\tilde{\Gamma}(W) \leq CW^2$ .*

(1) For any  $t > 0$  and  $f \in \mathcal{B}_b(M)$ ,

$$\begin{aligned} & \Gamma(P_t f) + \frac{\rho_2(e^{2\rho_1 t} - 1 - 2\rho_1 t)}{\rho_1(e^{2\rho_1 t} - 1)} \Gamma^Z(P_t f) \\ & \leq \left(1 + \frac{\kappa(e^{2\rho_1^+ t} - 1)^2}{\rho_2(e^{2\rho_1^+ t} - 1 - 2\rho_1^+ t)}\right) \frac{\rho_1}{e^{2\rho_1 t} - 1} \{P_t f^2 - (P_t f)^2\}, \end{aligned}$$

where when  $\rho_1 \leq 0$ ,

$$\frac{(e^{2\rho_1^+ t} - 1)^2}{e^{2\rho_1^+ t} - 1 - 2\rho_1^+ t} := \lim_{r \downarrow 0} \frac{(e^r - 1)^2}{e^r - 1 - r} = 2.$$

Consequently, if  $\rho_1 > 0$  and  $P_t$  is symmetric w.r.t. a probability measure  $\mu$ , then the Poincaré inequality

$$(3.11) \quad \mu(f^2) \leq \frac{1}{\rho_1} \mu(\Gamma(f)) + \mu(f)^2, \quad f \in C_0^1(M)$$

holds.

(2) If (1.7) holds, then for any  $t > 0$  and positive  $f \in \mathcal{B}_b(M)$ ,

$$\begin{aligned} & \Gamma(P_t f) + \frac{\rho_2(e^{2\rho_1 t} - 1 - 2\rho_1 t)}{\rho_1(e^{2\rho_1 t} - 1)} \Gamma^Z(P_t f) \\ & \leq \left(1 + \frac{\kappa(e^{2\rho_1^+ t} - 1)^2}{\rho_2(e^{2\rho_1^+ t} - 1 - 2\rho_1^+ t)}\right) \frac{2\rho_1(P_t f) \{P_t(f \log f) - (P_t f) \log P_t f\}}{e^{2\rho_1 t} - 1}. \end{aligned}$$

*Proof.* Let

$$\begin{aligned} b_0(s) &= \frac{e^{2\rho_1(t-s)} - 1}{2\rho_1}, \\ b_1(s) &= 2\rho_2 \int_s^t b_0(r) dr = \frac{\rho_2(e^{2\rho_1(t-s)} - 1 - 2\rho_1(t-s))}{2\rho_1^2}, \quad s \in [0, t]. \end{aligned}$$

Then it is easy to see that  $b_1 + 2b_0\rho_2 = 0$  and

$$\begin{aligned} \left\{b'_0 + 2b_0\left(\rho_1 - \frac{Kb_0}{b_1}\right)\right\}(s) &= -1 - \frac{\kappa(e^{2\rho_1(t-s)} - 1)^2}{\rho_2(e^{2\rho_1(t-s)} - 1 - 2\rho_1(t-s))} \\ &\geq -1 - \frac{\kappa(e^{2\rho_1^+ t} - 1)^2}{\rho_2(e^{2\rho_1^+ t} - 1 - 2\rho_1^+ t)}. \end{aligned}$$

Since (1.3) implies (1.5) for  $l = 1$ ,  $\Gamma^{(1)} = \Gamma^Z$ ,  $K_0(r) = \rho_1 - \frac{\kappa}{r}$  and  $K_1(r) = \rho_2$ , the desired derivative inequalities follows from (3.1) and (3.5).  $\square$

## 4 Examples

Additionally to those examples given in [7] such that (1.1) holds, we present three simple examples for which (1.3) (and hence (1.1)) is not available but our more general condition (1.5) holds true. For the first example the Poincaré and log-Sobolev inequalities are confirmed in the symmetric setting, while for the other two examples (1.7) does not hold so that we are only able to derive results in Proposition 3.1.

### 4.1 Example A

Let  $M = \mathbb{R} \times \bar{M}$ , where  $\bar{M}$  is a complete connected Riemannian manifold. Let  $\bar{L}$  be an elliptic differential operator on  $\bar{M}$  satisfying the curvature-dimension condition

$$(4.1) \quad \bar{\Gamma}_2(f) \geq K\bar{\Gamma}(f) + \frac{(\bar{L}f)^2}{m}, \quad f \in C^\infty(\bar{M}),$$

for some constant  $K \geq 0$  and  $m \in (1, \infty)$ , where  $\bar{\Gamma}$  is the square field of  $\bar{L}$  and  $\bar{\Gamma}_2$  is the associated curvature operator, i.e.  $\bar{\Gamma}_2(f) = \frac{1}{2}\bar{L}\bar{\Gamma}(f) - \bar{\Gamma}(f, Lf)$ . Consider

$$Lf(x, y) = f_{xx}(x, y) - r_0 x f_x(x, y) + x^2 \bar{L}f(x, \cdot)(y), \quad f \in C^\infty(M), (x, y) \in M$$

for some constant  $r_0 \in \mathbb{R}$ , where and in the sequel,  $f_{x_1 \dots x_k} := \frac{\partial^k}{\partial x_1 \dots \partial x_k} f$ ,  $k \geq 1$ . Then

$$\Gamma^{(0)}(f, g)(x, y) := \Gamma(f, g)(x, y) = (f_x g_x)(x, y) + x^2 \bar{\Gamma}(f(x, \cdot), g(x, \cdot))(y).$$

Let

$$\Gamma^{(1)}(f, g)(x, y) = \bar{\Gamma}(f(x, \cdot), g(x, \cdot))(y), \quad f \in C^\infty(M), (x, y) \in M.$$

According to (4.1), there exists a positive smooth compact function  $\bar{W}$  on  $\bar{M}$  such that  $\bar{L}\bar{W}, \bar{\Gamma}(\bar{W}) \leq 1$ . In fact, let  $\bar{\rho}$  be the intrinsic distance to a fixed point induced by  $\bar{\Gamma}$ , by (4.1) for  $K \geq 0$  and the comparison theorem, one has (see [13])

$$\bar{L}\bar{\rho} \leq \frac{m-1}{\bar{\rho}}$$

outside the fixed point and the cut-locus of this point. By Greene-Wu's approximation theorem (see [10]), we may assume that  $\bar{\rho}^2$  is smooth so that  $\bar{L}\sqrt{1 + \bar{\rho}^2} \leq c_1$  holds for some constant  $c_1 > 0$ . Noting that  $\bar{\Gamma}(\bar{\rho}) = 1$ , we may take  $\bar{W} = \varepsilon \sqrt{1 + \bar{\rho}^2}$  for small enough constant  $\varepsilon > 0$ .

Now, let  $W(x, y) = 1 + x^2 + \bar{W}(y)$ , which is a smooth compact function on  $M$ . It is easy to see that

$$(4.2) \quad LW(x, y) \leq 2(1 + r_0^-)W(x, y), \quad \tilde{\Gamma}(W)(x, y) = 4x^2 + (1 + x^2)\bar{\Gamma}(\bar{W})(y) \leq 5W(x, y),$$

where  $\tilde{\Gamma} = \Gamma + \Gamma^{(1)}$ .

**Proposition 4.1.** *The generalized curvature condition (1.5) holds for  $l = 1$  and*

$$K_1(r) = 1, \quad K_0(r) = \left(r_0 - \frac{m}{r}\right) \wedge \left(Kr - r_0 - \frac{4}{r}\right), \quad r > 0,$$

*and (1.7) holds. Consequently:*

(1) *Propositions 3.1 and 3.3 hold for  $b_0(0) = t, b_1(0) = t^2$  and*

$$c_b = 1 + 2 \sup_{r \in (0, t)} \left\{ (m - r_0 r) \vee (r_0 r + 4 - Kr^2) \right\}.$$

(2) *If  $K, r_0 > 0$  and  $\bar{L}$  is symmetric w.r.t. a probability measure  $\bar{\mu}$  on  $\bar{M}$ , then  $P_t$  is symmetric w.r.t.*

$$\mu(dx, dy) := \left( \frac{\sqrt{r_0} \exp[-\frac{r_0}{2}x^2]}{\sqrt{2\pi}} dx \right) \bar{\mu}(dy),$$

*and the Poincaré inequality (3.10) holds for*

$$\lambda = \min \left\{ \frac{2K}{r_0 + \sqrt{r_0^2 + 20K}}, \frac{r_0}{m+1} \right\} > 0.$$

*Moreover, the log-Sobolve inequality*

$$\mu(f^2 \log f^2) \leq c \mu(\Gamma(f)), \quad f \in C_0^1(M), \mu(f^2) = 1$$

*holds for some constant  $c > 0$ .*

*Proof.* (i) The proof of (1.7) is trivial. Below we intend to prove (1.5) for the desired  $K_0$  and  $K_1$ ; that is,

$$(4.3) \quad \Gamma_2(f) + r\Gamma_2^{(1)}(f) \geq \Gamma^{(1)}(f) + \left\{ \left(r_0 - \frac{m}{r}\right) \wedge \left(Kr - r_0 - \frac{4}{r}\right) \right\} \Gamma(f), \quad f \in C^\infty(M).$$

It is easy to see that at point  $(x, y)$ ,

$$\begin{aligned} \Gamma_2(f) &= f_{xx}^2 + (1 - r_0 x^2) \Gamma^{(1)}(f) + 4x \Gamma^{(1)}(f, f_x) \\ &\quad + 2x^2 \Gamma^{(1)}(f_x) + x^4 \bar{\Gamma}_2(f(x, \cdot))(y) - 2x f_x \bar{L} f(x, \cdot)(y) + r_0 f_x^2, \\ \Gamma_2^{(1)}(f) &= x^2 \bar{\Gamma}_2(f(x, \cdot))(y) + \Gamma^{(1)}(f_x). \end{aligned}$$

Combining these with (4.1) we obtain

$$\begin{aligned} &\Gamma_2(f) + r\Gamma_2^{(1)}(f) \\ &\geq \Gamma^{(1)}(f) - r_0 x^2 \Gamma^{(1)}(f) + \left\{ (2x^2 + r) \Gamma^{(1)}(f_x) + 4x \Gamma^{(1)}(f, f_x) \right\} + (x^4 + r x^2) K \Gamma^{(1)}(f) \\ &\quad + \left\{ \frac{(x^4 + r x^2) (\bar{L} f(x, \cdot)(y))^2}{m} - 2x f_x \bar{L} f(x, \cdot)(y) \right\} + r_0 f_x^2 \\ &\geq \Gamma^{(1)}(f) + \left( K(x^2 + r) - r_0 - \frac{4}{2x^2 + r} \right) x^2 \Gamma^{(1)}(f) - \frac{m x^2}{x^4 + r x^2} f_x^2 + r_0 f_x^2 \\ &\geq \Gamma^{(1)}(f) + \left( Kr - r_0 - \frac{4}{r} \right) x^2 \Gamma^{(1)}(f) + \left( r_0 - \frac{m}{r} \right) f_x^2 \\ &\geq \Gamma^{(1)}(f) + \left\{ \left(r_0 - \frac{m}{r}\right) \wedge \left(Kr - r_0 - \frac{4}{r}\right) \right\} \Gamma(f). \end{aligned}$$

Therefore, (4.3) holds.

(ii) Whence (1.7) and (1.5) are confirmed for the desired  $K_0$  and  $K_1$ , due to (4.2) the assumption **(A)** holds. Then (1) follows immediately by taking

$$b_0(s) = t - s, \quad b_1(s) = (t - s)^2, \quad s \in [0, t].$$

It remains to prove the the Poincaré inequality and the log-Sobolev inequality for  $K, r_0 > 0$  in the symmetric setting. By Proposition 3.5, the Poincaré inequality holds for

$$\lambda = \sup_{r>0} \left\{ K_0(r) \wedge \frac{K_1(r)}{r} \right\} = \sup_{r>0} \left\{ \frac{1}{r} \wedge \left( r_0 - \frac{m}{r} \right) \wedge \left( Kr - r_0 - \frac{4}{r} \right) \right\}.$$

Since  $\frac{1}{r}$  is decreasing in  $r > 0$  with range  $(0, \infty)$  while  $\left( r_0 - \frac{m}{r} \right) \wedge \left( Kr - r_0 - \frac{4}{r} \right)$  is increasing in  $r > 0$  with range  $(-\infty, r_0)$ ,  $\lambda$  is reached by a unique number  $r_1 > 0$  such that

$$\frac{1}{r_1} = \left( r_0 - \frac{m}{r_1} \right) \wedge \left( Kr_1 - r_0 - \frac{4}{r_1} \right).$$

Then the value of  $\lambda$  can be fixed by considering the following two situations:

- A. If  $r_0 - \frac{m}{r_1} \leq Kr_1 - r_0 - \frac{4}{r_1}$ , we have  $\frac{1}{r_1} = r_0 - \frac{m}{r_1}$  so that  $r_1 = \frac{m+1}{r_0}$  and hence,  $\lambda = \frac{r_0}{m+1}$ .
- B. If  $Kr_1 - r_0 - \frac{4}{r_1} < r_0 - \frac{m}{r_1}$ , then  $\frac{1}{r_1} = Kr_1 - r_0 - \frac{4}{r_1}$  so that  $r_1 = \frac{r_0 + \sqrt{r_0^2 + 20K}}{2K}$  and  $\lambda = \frac{2K}{r_0 + \sqrt{r_0^2 + 20K}}$ .

To prove the validity of the log-Sobolev inequality, we observe that

$$c_b \leq 1 + 2 \left\{ m \wedge \left( \frac{r_0^2}{4K} + 4 \right) \right\} =: c_0.$$

Moreover, by the Meyer diameter theorem (see [4] and references within), (4.1) with  $K > 0$  implies that the intrinsic distance induced by  $\bar{\Gamma}$  is bounded by a constant  $D > 0$ . Noting that

$$\Gamma_b(f)(x, y) = t f_x^2(x, y) + t^2 \bar{\Gamma}((x, \cdot))(y),$$

we conclude that

$$\rho_b^2((x, y), (x', y')) \leq \frac{|x - x'|^2}{t} + \frac{D^2}{t^2}, \quad t > 0, (x, y), (x', y') \in M.$$

Thus, for any  $\lambda > \frac{c_0}{4} \left( \geq \frac{c_b}{4} \right)$ ,  $\mu(e^{\lambda \rho_b^2(o, \cdot)^2}) < \infty$  holds for  $o \in M$  and large  $t > 0$ . Therefore, as indicated in the end of Subsection 3.2, the log-Sobolev inequality is valid.  $\square$

## 4.2 Example B

Consider the Grushin operator  $L = f_{xx} + x^{2l}f_{yy}$  on  $M := \mathbb{R}^2$ , where  $l \in \mathbb{N}$ . We have

$$\Gamma^{(0)}(f, g)(x, y) := \Gamma(f, g)(x, y) = (f_x g_x)(x, y) + x^{2l}(f_y g_y)(x, y)$$

and  $L = X^2 + Y^2$  for  $X = \frac{\partial}{\partial x}, Y = x^l \frac{\partial}{\partial y}$ . When  $l \geq 2$ ,  $\{X, Y, \nabla_X Y = lx^{l-1} \frac{\partial}{\partial y}, \nabla_Y X = 0\}$  does not span the whole space for  $x = 0$ . So, as explained in point (a) in the Introduction, (1.3) is invalid. Let

$$\Gamma^{(i)}(f, g)(x, y) = x^{2(l-i)}(f_y g_y)(x, y), \quad 1 \leq i \leq l.$$

It is easy to see that  $W(x, y) := 1 + x^2 + \frac{y^2}{1+x^{2l}}$  is a smooth compact function such that

$$(4.4) \quad LW \leq CW, \quad \tilde{\Gamma}(W) \leq CW^2$$

holds for some constant  $C > 0$ .

**Proposition 4.2.** *There exist two constants  $\alpha, \beta > 0$  depending only on  $l$  such that (1.5) holds for*

$$K_0(r_1, \dots, r_l) = -\alpha \sum_{i=1}^l \frac{r_{i-1}^{i-1}}{r_i^i}, \quad K_i(r_1, \dots, r_l) = \beta r_{i-1}, \quad 1 \leq i \leq l, r_0 = 1, r_i > 0.$$

Consequently, Proposition 3.1 holds for  $b_i(0) = c_i t^{2l-1+i}$ , where

$$c_0 = 1, \quad c_i = \frac{2\beta c_{i-1}}{2l-1+i}, \quad 1 \leq i \leq l,$$

and

$$c_b = \sup_{r \in (0, t)} \left\{ (2l-1)r^{2(l-1)} + 2\alpha \sum_{i=1}^l \frac{c_{i-1}^{i-1}}{c_i^i} r^{2(l-i)} \right\} \leq C_0(1+t^{l-1}), \quad t > 0$$

for some constant  $C_0 > 0$ .

*Proof.* According to Proposition 3.1 for  $b_i(s) = c_i(t-s)^{2l-1+i}$ ,  $s \in [0, t], 0 \leq i \leq l$ , it suffices to verify (1.5) for the desired  $\{K_i\}_{0 \leq i \leq l}$ , which satisfy

$$b'_i(s) + 2b_0(s)K_i\left(\frac{b_1}{b_0}, \dots, \frac{b_l}{b_0}\right)(s) = 0, \quad 1 \leq i \leq l, s \in [0, t]$$

and

$$b'_0(s) + 2b_0(s)K_0\left(\frac{b_1}{b_0}, \dots, \frac{b_l}{b_0}\right)(s) = (2l-1)(t-s)^{2(l-1)} + 2\alpha \sum_{i=1}^l \frac{c_{i-1}^{i-1}}{c_i^i} (t-s)^{2(l-i)}, \quad s \in [0, t].$$



It is easy to see that at point  $(x, y) \in \mathbb{R}^2$  and for  $1 \leq i \leq l$ ,

$$\begin{aligned}\Gamma_2(f) &= f_{xx}^2 + l(2l-1)x^{2(l-1)}f_y^2 + x^{4l}f_{yy}^2 + 2x^{2l}f_{xy}^2 + 4lx^{2l-1}f_yf_{xy} - 2lx^{2l-1}f_xf_{yy}, \\ \Gamma_2^{(i)}(f) &= (l-i)(2l-2i-1)x^{2(l-i-1)}f_y^2 + 4(l-i)x^{2l-2i-1}f_yf_{xy} + x^{2(l-i)}f_{xy}^2 + x^{2(2l-i)}f_{yy}^2.\end{aligned}$$

So, for  $r_0 = 1$  and  $r_i > 0, 1 \leq i \leq l$ ,

$$\begin{aligned}\Gamma_2(f) &+ \sum_{i=1}^l r_i \Gamma_2^{(i)}(f) \\ &\geq f_y^2 \sum_{i=0}^l r_i (l-i)(2l-2i-1)x^{2(l-i-1)} + f_{yy}^2 \sum_{i=0}^l r_i x^{2(2l-i)} - 2lx^{2l-1}f_xf_{yy} \\ &\quad + f_{xy}^2 \left\{ 2x^{2l} + \sum_{i=1}^l r_i x^{2(l-i)} \right\} + f_yf_{xy} \sum_{i=0}^l 4r_i(l-i)x^{2l-2i-1} \\ &\geq f_y^2 \sum_{i=1}^l r_{i-1}(l+1-i)(2l-2i+1)x^{2(l-i)} - \frac{l^2}{r_1}f_x^2 - 4 \sum_{i=0}^{l-1} \frac{r_i^2(l-i)^2x^{4l-4i-2}}{r_{i+1}x^{2(l-i-1)}}f_y^2 \\ &\geq \frac{f_y^2}{2} \sum_{i=1}^l r_{i-1}(l+1-i)(2l-2i+1)x^{2(l-i)} - \frac{l^2}{r_1}f_x^2 \\ &\quad - \frac{1}{2} \sum_{i=1}^l \left\{ \frac{8r_{i-1}^2(l+1-i)^2x^{2(l+1-i)}}{r_i} - r_{i-1}(l+1-i)(2l-2i+1)x^{2(l-i)} \right\} f_y^2 \\ &\geq \frac{1}{2} \sum_{i=1}^l r_{i-1}(l+1-i)(2l-2i+1)x^{2(l-i)}\Gamma^{(i)}(f) - \frac{l^2}{r_1}f_x^2 - \sum_{i=1}^l \frac{\alpha_i r_{i-1}^{i-1}}{r_i^i} x^{2l} f_y^2\end{aligned}$$

holds for some constants  $\alpha_i > 0, 1 \leq i \leq l$ , where the last step is due to the fact that for constants  $A_i, B_i > 0$ ,

$$A_i x^{-2(i-1)} - B_i x^{-2i} \leq \sup_{s>0} \{A_i s^{i-1} - B_i s^i\} = \frac{(i-1)^{i-1} A_i^i}{i^i B_i^{i-1}}.$$

□

### 4.3 Example C

Consider  $Lf = f_{xx} + x^2 f_{yy} + y^2 f_{zz} = X_1^2 + X_2^2 + X_3^2$  on  $\mathbb{R}^3$ , where  $X_1 = \frac{\partial}{\partial x}$ ,  $X_2 = x \frac{\partial}{\partial y}$ ,  $X_3 = y \frac{\partial}{\partial z}$ . It is easy to see that  $\{X_i, \nabla_{X_i} X_j\}_{1 \leq i \leq 3}$  does not span  $\mathbb{R}^3$  when  $x = 0$ , so that (1.3) is not available according to observation (a) in Introduction. We have

$$\Gamma^{(0)}(f, g)(x, y, z) := \Gamma(f, g)(x, y, z) = (f_x g_x)(x, y, z) + x^2 (f_y g_y)(x, y, z) + y^2 (f_z g_z)(x, y, z).$$

Let

$$\Gamma^{(1)}(f, g)(x, y, z) = (f_y g_y)(x, y, z) + x^2 (f_z g_z)(x, y, z), \quad \Gamma^{(2)}(f, g)(x, y, z) = (f_z g_z)(x, y, z).$$

It is easy to see that  $W(x, y, z) := 1 + x^2 + y^2$  is a smooth compact function on  $\mathbb{R}^3$  such that (4.4) holds for some constant  $C > 0$ .

**Proposition 4.3.** (1.5) holds for

$$K_0(r_1, r_2) = -\left(\frac{5}{r_1} + \frac{2r_1}{r_2}\right), \quad K_1(r_1, r_2) = 1 - \frac{4r_1^2}{r_2}, \quad K_2(r_1, r_2) = r_1, \quad r_1, r_2 > 0.$$

Consequently, Proposition 3.1 holds for

$$b_0(0) = t, \quad b_1(0) = \frac{t^2}{7}, \quad b_2(0) = \frac{2t^3}{21},$$

and  $c_b = 77$ .

*Proof.* We first prove (1.5) for the desired  $K_i, 0 \leq i \leq 2$ . It is easy to see that at point  $(x, y, z)$ ,

$$\begin{aligned} \Gamma_2(f) &= f_{xx}^2 + f_y^2 + x^2 f_z^2 + 2x^2 f_{xy}^2 + 2y^2 f_{xz}^2 + x^4 f_{yy}^2 + 2x^2 y^2 f_{yz}^2 + y^4 f_{zz}^2 \\ &\quad + 4x f_y f_{xy} + 4x^2 y f_z f_{yz} - 2x f_x f_{yy} - 2x^2 y f_y f_{zz}, \\ \Gamma_2^{(1)}(f) &= f_z^2 + f_{xy}^2 + x^2 f_{yy}^2 + (y^2 + x^4) f_{yz}^2 + x^2 f_{xz}^2 + x^2 y^2 f_{zz}^2 + 4x f_z f_{xz} - 2y f_y f_{zz}, \\ \Gamma_2^{(2)}(f) &= f_{xz}^2 + x^2 f_{yz}^2 + y^2 f_{zz}^2. \end{aligned}$$

Therefore,

$$\begin{aligned} &\Gamma_2(f) + r_1 \Gamma_2^{(1)}(f) + r_2 \Gamma_2^{(2)}(f) \\ &\geq \Gamma^{(1)}(f) + r_1 \Gamma^{(2)}(f) + \{(2x^2 + r_1) f_{xy}^2 + 4x f_y f_{xy}\} + \{(2y^2 + r_1 x^2 + r_2) f_{xz}^2 + 4r_1 x f_z f_{xz}\} \\ &\quad + \{(x^4 + r_1 x^2) f_{yy}^2 - 2x f_x f_{yy}\} + \{(2x^2 y^2 + r_1 y^2 + r_1 x^4 + r_2 x^2) f_{yz}^2 + 4x^2 y f_z f_{yz}\} \\ &\quad + \{(y^4 + r_1 x^2 y^2 + r_2 y^2) f_{zz}^2 - 2x^2 y f_y f_{zz} - 2r_1 y f_y f_{zz}\} \\ &\geq \Gamma^{(1)}(f) + r_1 \Gamma^{(2)}(f) - \frac{4x^2}{2x^2 + r_1} f_y^2 - \frac{4r_1^2 x^2}{r_2 + r_1 x^2 + 2y^2} f_z^2 - \frac{x^2}{x^4 + r_1 x^2} f_x^2 \\ &\quad - \frac{4x^4 y^2}{2x^2 y^2 + r_1 y^2 + r_1 x^4 + r_2 x^2} f_z^2 - \frac{(r_1 + x^2)^2}{y^2 + r_1 x^2 + r_2} f_y^2 \\ &\geq \Gamma^{(1)}(f) + r_1 \Gamma^{(2)}(f) - \frac{4x^2}{r_1} f_y^2 - \frac{4r_1^2 x^2}{r_2} f_z^2 - \frac{1}{r_1} f_x^2 - \frac{4y^2}{r_1} f_z^2 - \frac{x^2}{r_1} f_y^2 - \frac{2r_1 x^2}{r_2} f_y^2 - \frac{r_1^2}{r_2} f_y^2 \\ &\geq \left(1 - \frac{4r_1^2}{r_2}\right) \Gamma^{(1)}(f) + r_1 \Gamma^{(2)}(f) - \left(\frac{5}{r_1} + \frac{2r_1}{r_2}\right) \Gamma(f). \end{aligned}$$

This implies (1.5) for the claimed  $K_i, 0 \leq i \leq 2$ .

Next, take

$$b_0(s) = t - s, \quad b_1(s) = \frac{1}{7}(t - s)^2, \quad b_2(s) = \frac{2}{21}(t - s)^3, \quad s \in [0, t].$$

Then

$$\begin{aligned}\left\{b'_1 + 2b_0K_1\left(\frac{b_1}{b_0}, \frac{b_2}{b_0}\right)\right\}(s) &= -\frac{2(t-s)}{7} + 2(t-s)\left(1 - \frac{6}{7}\right) = 0, \\ \left\{b'_2 + 2b_0K_2\left(\frac{b_1}{b_0}, \frac{b_2}{b_0}\right)\right\}(s) &= -\frac{6(t-s)}{21} + \frac{2(t-s)^2}{7} = 0, \\ \left\{b'_0 + 2b_0K_0\left(\frac{b_1}{b_0}, \frac{b_2}{b_0}\right)\right\}(s) &= -1 - 2(t-s)\left(\frac{5}{t-s} + \frac{3}{t-s}\right) = -77.\end{aligned}$$

Therefore, the second assertion holds.  $\square$

## 5 An extension of Theorem 1.1

As mentioned in the end of Section 1, for some highly degenerate subelliptic operators (1.5) is only available for non-positively definite differential forms  $\Gamma^{(i)}$  and for some  $r_1, \dots, r_l > 0$ . For instance, when  $L = \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial y}$ , one has  $\Gamma(f) = f_x^2$  and  $\Gamma_2(f) = f_{xx}^2 - f_x f_y$ . So, to verify (1.5), it is natural to take  $\Gamma^{(1)}(f, g) = -\frac{f_x g_y + f_y g_x}{2}$ , which is however not positively definite. See Example 5.1 below for details.

To investigate such operators, we make use of the following weaker version of assumption **(A)**. We call bilinear symmetric form  $\bar{\Gamma} : C^3(M) \times C^3(M) \rightarrow C^2(M)$  a  $C^2$  symmetric differential form, if

$$\bar{\Gamma}(fg, h) = f\bar{\Gamma}(g, h) + g\bar{\Gamma}(f, h), \quad \bar{\Gamma}(f, \phi \circ g) = (\phi' \circ g)\bar{\Gamma}(f, g), \quad f, g \in C^3(M), \phi \in C^1(\mathbb{R})$$

holds.

**(B)** *There exist some  $C^2$  symmetric differential form  $\{\Gamma^{(i)}\}_{1 \leq i \leq l}$ , a non-empty set  $\Omega \subset (0, \infty)^l$ , a smooth compact function  $W \geq 1$ , and some function  $\{K_i\}_{0 \leq i \leq l} \subset C(\Omega)$  such that*

$$(B1) \quad \Gamma_2 + \sum_{i=1}^l r_i \Gamma_2^{(i)} \geq \sum_{i=0}^l K_i(r_1, \dots, r_l) \Gamma^{(i)}, \quad (r_1, \dots, r_l) \in \Omega, \text{ where } \Gamma^{(0)} = \Gamma.$$

$$(B2) \quad LW \leq CW \text{ and } \sum_{i=0}^l |\Gamma^{(i)}(W)| \leq CW^2 \text{ hold for some constant } C > 0.$$

(B3) *There exist  $\varepsilon > 0$  and  $\tilde{r} = (\tilde{r}_1, \dots, \tilde{r}_l) \in \Omega$  such that*

$$\tilde{\Gamma} := \Gamma + \sum_{i=1}^l \tilde{r}_i \Gamma^{(i)} \geq \varepsilon \sum_{i=0}^l |\Gamma^{(i)}|.$$

**Theorem 5.1.** *Assume **(B)**. For fixed  $t > 0$ , let  $\{b_i\}_{0 \leq i \leq l} \subset C^1([0, t])$  be strictly positive on  $(0, t)$  such that*

$$(i) \quad \left(\frac{b_1}{b_0}, \dots, \frac{b_l}{b_0}\right)(s) \in \Omega \text{ holds for all } s \in (0, t);$$

$$(ii) \quad b'_i(s) + 2\left\{b_0K_i\left(\frac{b_1}{b_0}, \dots, \frac{b_l}{b_0}\right)\right\}(s) = 0, \quad s \in (0, t), 1 \leq i \leq l.$$

Then assertions in (1) and (2) of Theorem 1.1 hold.

*Proof.* By (B1) and (B3),  $\tilde{\Gamma}_2 \geq K\tilde{\Gamma}$  and  $\tilde{\Gamma} \geq \varepsilon \sum_{i=0}^l |\Gamma^{(i)}|$  hold for some  $K \in \mathbb{R}$  and  $\varepsilon > 0$ . Combining these with (B2) and repeating the proof of Lemma 2.1, we conclude that  $\{\Gamma^{(i)}(P.f)\}_{0 \leq i \leq l}$  are bounded on  $[0, t] \times M$ . Therefore, due to (i) and (ii) the proof of Theorem 1.1 works also for the present case.  $\square$

To illustrate this result, we consider the following example mentioned in the beginning of this section, where the resulting gradient and Hanrck inequalities have the same time behaviors as the corresponding ones presented in [11, Corollaries 3.2 and 4.2] by using coupling methods. In this example, it is easy to find correct choices of  $W$ ,  $\Gamma^{(i)}$ ,  $K_i$  and  $\Omega$  such that assumption **(B)** and condition (i) in Theorem 5.1 hold. The technical (also difficult) point is to construct functions  $\{b_i\}_{i=0}^l$  such that condition (ii) holds and  $\sum_{i=0}^l b_i(0)\Gamma^{(i)}$  is an elliptic square field.

**Example 5.1.** Consider  $L = \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial y}$  on  $\mathbb{R}^2$ . We have

$$\Gamma^{(0)}(f, g) := \Gamma(f, g) = f_x g_s.$$

Let

$$\Gamma^{(1)}(f, g) = -\frac{1}{2}(f_x g_y + f_y g_x), \quad \Gamma^{(2)} = f_x g_y.$$

Then **(B)** holds for  $W(x, y) = 1 + x^2 + y^2$ ,  $\Omega = \{(r_1, r_2) : r_1, r_2 > 0, r_1^2 \leq 4r_2\}$ , and

$$K_0(r_1, r_2) = 0, \quad K_1(r_1, r_2) = 1 + r_2, \quad K_2(r_1, r_2) = \frac{r_1}{2}.$$

Moreover, (1.7) holds. Consequently, for any  $\theta \in (\frac{3}{2}, 2)$ , letting  $t_\theta > 0$  be the unique solution

$$(\coth[\sqrt{2}t_\theta] - 1)^2 = 2\left(\sqrt{2}\sinh[\sqrt{2}t_\theta] - 2t_\theta^2\right),$$

there holds

$$(5.1) \quad \frac{2-\theta}{2} \left\{ (t \wedge t_\theta)(P_t f)_x^2 + \frac{(t \wedge t_\theta)^3}{3} (P_t f)_y^2 \right\} \leq (P_t f) \{ P_t(f \log f) - (P_t f) \log P_t f \}, \quad t > 0$$

so that the Harnack inequality

$$(5.2) \quad (P_t f)^\alpha((x, y)) \leq (P_t f^\alpha(x, y)) \exp \left[ \frac{\alpha}{2(2-\theta)(\alpha-1)} \left( \frac{|x-x'|^2}{t \wedge t_\theta} + \frac{3|y-y'|^2}{(t \wedge t_\theta)^3} \right) \right]$$

holds for all  $\alpha > 1, t > 0, (x, y), (x', y') \in \mathbb{R}^2$  and positive  $f \in \mathcal{B}_b(\mathbb{R}^2)$ .

*Proof.* Obviously, (B2) holds for the given  $W$  and (B3) holds for  $r_1 = r_2 = 1$  (hence  $(r_1, r_2) \in \Omega$ ) and  $\varepsilon = \frac{1}{4}$ . Next, it is easy to see that (1.7) holds and

$$\Gamma_2(f) = f_{xx}^2 - f_x f_y, \quad \Gamma_2^{(1)}(f) = \frac{1}{2} f_y^2 - f_{xx} f_{xy}, \quad \Gamma^{(2)}(f) = f_{xy}^2 - f_x f_y.$$

Then, for  $r_1, r_2 > 0$  with  $r_1^2 \leq 4r_2$ , we have

$$\Gamma_2(f) + r_1 \Gamma_2^{(1)}(f) + r_2 \Gamma_2^{(2)}(f) \geq (1 + r_2) \Gamma^{(1)}(f) + \frac{r_1}{2} \Gamma^{(2)}.$$

Therefore, (B1) holds.

To prove (5.1) and (5.2), we take  $l = 2$  and

$$b_0(s) = t - s, \quad b_1(s) = \coth[\sqrt{2}(t - s)] - 1, \quad b_3(s) = \frac{1}{2} \sinh[\sqrt{2}(t - s)] - (t - s), \quad s \in [0, t].$$

Then it is easy to see that (ii) holds. Observing that the function

$$\psi(r) := \frac{(\coth r - 1)^2}{r \sinh r - r^2}, \quad r > 0$$

is increasing with  $\lim_{r \rightarrow \infty} \psi(r) = \infty$  and  $\lim_{r \rightarrow 0} \psi(r) = \frac{3}{2}$ , for any  $\theta \in (\frac{3}{2}, 2)$  the claimed quantity  $t_\theta$  exists uniquely and for any  $s \leq t_\theta$ ,

$$(5.3) \quad \frac{b_1^2(s)}{b_0^2(s)} \leq \frac{\theta^2 b_2(s)}{b_0(s)} \leq \frac{4b_2(s)}{b_0(s)}.$$

Thus,  $(\frac{b_1}{b_0}, \frac{b_2}{b_0}) \in \Omega$  holds on  $[0, t]$  provided  $t \leq t_\theta$ . Since the right-hand side of (5.1) is increasing in  $t$ , we may and do assume that  $t \in (0, t_\theta]$ . Noting that  $b_0(0) = 1$  and  $b_i(t) = 0$  for  $0 \leq i \leq 2$ , it follows from Theorem 5.1 and (5.3) that

$$\begin{aligned} (P_t f) \{P_t(f \log f) - (P_t f) \log P_t f\} &\geq \sum_{i=0}^2 b_i(0) \Gamma^{(i)}(P_t f) \\ &= \frac{(2 - \theta)b_0(0)}{2} \left\{ (P_t f)_x^2 + \frac{b_2(0)}{b_0(0)} (P_t f)_y^2 \right\} + b_0(0) \left\{ \frac{\theta}{2} (P_t f)_x^2 - \frac{b_1(0)}{b_0(0)} (P_t f)_x (P_t f)_y + \frac{\theta b_2(0)}{2b_0(0)} (P_t f)_y^2 \right\} \\ &\geq \frac{2 - \theta}{2} \left\{ t (P_t f)_x^2 + \left( \frac{\sinh[\sqrt{2}t]}{\sqrt{2}} - t \right) (P_t f)_y^2 \right\} \geq \frac{2 - \theta}{2} \left\{ t (P_t f)_x^2 + \frac{t^3}{3} (P_t f)_y^2 \right\}. \end{aligned}$$

Therefore, (5.1) holds.

Finally, letting  $\bar{\rho}$  be the intrinsic distance induced by the square field

$$\bar{\Gamma}(f) := \frac{2 - \theta}{2} \left\{ (t \wedge t_\theta) (P_t f)_x^2 + \frac{(t \wedge t_\theta)^3}{3} (P_t f)_y^2 \right\},$$

we have

$$\bar{\rho}((x, y), (x', y'))^2 = \frac{2}{2 - \theta} \left( \frac{|x - x'|^2}{t \wedge t_\theta} + \frac{3|y - y'|^2}{(t \wedge t_\theta)^3} \right).$$

Then the desired Harnack inequality follows from Lemma 3.4 for  $\gamma(\delta) = \frac{1}{4\delta}$  since (5.1) is equivalent to

$$\sqrt{\bar{\Gamma}(P_t f)} \leq \delta \{P_t(f \log f) - (P_t f) \log P_t f\} + \frac{P_t f}{4\delta}, \quad \delta > 0.$$

□

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